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Countable compactness and finite powers of topological groups without convergent sequences ☆

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Abstract

We show under $\mathrm{MA}_{\mathrm{countable}}$ that for every positive integer n there exists a topological group G without non-trivial convergent sequences such that G^n is countably compact but G^{n+1} is not. This answers the finite case of Comfort's Question 477 in the Open Problems in Topology. We also show under $\mathrm{MA}_{\mathrm{countable}} + 2^{<\mathfrak{c}} = \mathfrak{c}$ that there are $2^{\mathfrak{c}}$ non-homeomorphic group topologies as above if $n \geqslant 2$. We apply the construction on suitable sets, answering the finite case of a question of D. Dikranjan on the productivity of suitability and in a topological game defined by Bouziad.

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1. Introduction

One of the best well-known results in topology is Tychonoff's theorem on the productivity of compactness. A natural question is whether other compact-like properties are productive as well. There are countably compact spaces whose square are not even pseudocompact (due independently to Novák [21] and Terasaka [25]). Frolík [11] extended this study to finite products, showing that for every n there exists a space X such that

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X^n is countably compact but X^{n+1} is not. On the other hand, Scarborough and Stone [24] showed that if the $2^{2^{\mathfrak{c}}}$th power of a space is countably compact then every power is countably compact. This last result was improved by Ginsburg and Saks [14] who showed that a space is countably compact for every power if and only if its $2^{\mathfrak{c}}$th power is countably compact. The natural question whether $2^{\mathfrak{c}}$ was the best possible was answered under MA by Saks [23]. Recently, using an equivalence of Yang [30], M. Hrusak noted that in Shelah's model [2], the best possible cardinal in Ginsburg and Saks theorem is not larger than $\mathfrak{c}$.

Topological groups is an important class of spaces in which productivity has been studied, resulting in interesting theorems and examples. A quite unexpected result was obtained by Comfort and Ross [6], who showed that pseudocompactness becomes productive for the class of topological groups. A natural question, asked by Comfort, was whether the same would be true for countably compact topological groups. A consistent negative answer was obtained by van Douwen [10] who showed under MA that there exist two countably compact topological groups whose product is not countably compact. A decade later, Hart and van Mill [17] showed that there exists under $\mathrm{MA}_{\mathrm{countable}}$ a countably compact topological group whose square is not countably compact.

Because of these results, Comfort asked the following question in the Open Problems in Topology:

Question 1.1 [4]. *Is there, for every* (*not necessarily infinite*) *cardinal number $\alpha \leqslant 2^{\mathfrak{c}}$, a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact*?

Under $\mathrm{MA}_{\mathrm{countable}}$, it was shown that two [17] and three [28] are such cardinals. Furthermore, there are infinitely many such natural numbers [27]. All the examples above contain convergent sequences.

We answer Comfort's question for the finite case, under $\mathrm{MA}_{\mathrm{countable}}$, providing witnesses without non-trivial convergent sequences. In addition, if $2^{<\mathfrak{c}} = \mathfrak{c}$ is also assumed, then $2^{\mathfrak{c}}$ non-homeomorphic examples can be obtained for $n \geqslant 3$. Some kind of 'basis property' for sequences of length at most n are used to obtain the construction. The motivation for this approach came from Steve Watson's lecture on Kunen's solution to a question of van Douwen on Bohr topologies [20], given at University of São Paulo.

In [7], Dikranjan studied cardinal numbers related to the productivity of a topological property, restating Comfort's question in this setting. He also asked about the productivity of suitability, which was introduced by Hofmann and Morris. This concept, introduced in [18], is a natural generalization of monotheticity.

Definition 1.2. A subset S of a topological group H is a (closed) suitable set for H if (a) S is discrete in H and (b) S is closed in $H \setminus \{e\}$ (respectively H) and (c) the group generated by S is dense in H.

A topological group is monothetic if it has a suitable set of size 1. Hofmann and Morris showed that every locally compact group has a suitable set; in [5] the study of the non-locally compact case was started and it was followed in [8,9,22,26,29]. It is easy to see that if a power of H has a suitable set then every larger power also does. Dikranjan [7] asked then for what cardinal numbers κ, there exists a group G which does not have a suitable set for G^λ if $\lambda < \kappa$ but G^κ has a suitable set.

We show that under $\mathrm{MA}_{\text{countable}}$, Dikranjan's question has an affirmative answer for every positive integer.

In the last section we present the $\mathcal{G}$-spaces, defined through a game introduced by Bouziad [3] and show that under $\mathrm{MA}_{\text{countable}}$ there is, for each positive integer n, a topological group G such that G^n is a $\mathcal{G}$-space but G^{n+1} is not.

2. The examples and some preliminaries

Let G be the algebraic sum of $\mathfrak{c}$ copies of 2. We will topologize G to obtain our desired topological groups. The relation between suitability and countably compact groups without non-trivial convergent sequences was first noticed in [5]. We further explore this relation in finite products.

Lemma 2.1. *Fix $n \in \omega$. Suppose that $\{x_{m,j}\colon m < \omega \wedge j < n+2\}$ is a subset of a topological group $H \subseteq 2^{\mathfrak{c}}$ such that $\{(x_{m,n}, x_{m,n+1})\colon m \in \omega\}$ is dense in $H \times H$ and $T = \{(x_{m,j})_{j<n+1}\colon m \in \omega\} \cup \{(x_{m,j})_{j \in n\cup\{n+1\}}\colon m \in \omega\}$ is closed and discrete in H^{n+1}. Then H^{n+1} has a closed suitable set. If $H \subseteq 2^{\mathfrak{c}}$ has no non-trivial convergent sequences and H^n is countably compact then H^n has no suitable set.*

Proof. Let S be the set of $(n+1)$-uples that are permutations of elements of T, that is, $\{y_j\}_{j<n+1} \in S$ if there exists $m \in \omega$ and a bijection σ from $n+1$ into $n+1$ or $n \cup \{n+1\}$ such that $y_j = x_{m,\sigma(j)}$ for each $j < n+1$. Then clearly S is closed discrete and the group generated by S contains all the $n+1$-uples that are 0 in all coordinates but one and the remaining coordinate is of the form $x_{m,n} + x_{m,n+1}$ and $\{x_{m,n} + x_{m,n+1}\colon m \in \omega\}$ is dense in H. Therefore, the group generated by S is dense in H^{n+1} and S is a closed suitable set.

The group H^n does not have a suitable set since it is a countably compact group of order 2 without non-trivial convergent sequences (see [5]). $\square$

Example 2.2. ($\mathrm{MA}_{\text{countable}} + 2^{<\mathfrak{c}} = \mathfrak{c}$) For every positive integer n there exists a family of topological group topologies $\{\mathcal{T}_\alpha\colon \alpha < 2^{\mathfrak{c}}\}$ on the group G such that $\langle G, \mathcal{T}_\alpha\rangle \times \langle G, \mathcal{T}_\beta\rangle$ is not countably compact for any pair $\alpha < \beta < 2^{\mathfrak{c}}$, $\langle G, \mathcal{T}_\alpha\rangle^n$ is countably compact without non-trivial convergent sequences and $\langle G, \mathcal{T}_\alpha\rangle^{n+1}$ has a closed suitable set.

We will construct $2^{\mathfrak{c}}$ many topologies using the tree $2^{\leqslant\mathfrak{c}}$ assuming $\mathrm{MA}_{\text{countable}} + 2^{<\mathfrak{c}} = \mathfrak{c}$, but from the construction, it is clear that one can construct the example below using the tree $2^{\leqslant\omega}$.

Example 2.3. ($\mathrm{MA}_{\text{countable}}$) For every positive integer n there exists a family of topological group topologies $\{\mathcal{T}_\alpha\colon \alpha < \mathfrak{c}\}$ on the group G such that $\langle G, \mathcal{T}_\alpha\rangle \times \langle G, \mathcal{T}_\beta\rangle$ is not countably compact for any pair $\alpha < \beta < \mathfrak{c}$, $\langle G, \mathcal{T}_\alpha\rangle^n$ is countably compact and does not have a suitable set; and $\langle G, \mathcal{T}_\alpha\rangle^{n+1}$ is not countably compact and has a closed suitable set.

The following is an easy consequence of Example 2.2:

Example 2.4. Assume ($\mathrm{MA}_{\text{countable}} + 2^{<\mathfrak{c}} = \mathfrak{c}$). If $n \geqslant 2$ then there exists $2^{\mathfrak{c}}$ many non-homeomorphic group topologies on the Boolean group of size $\mathfrak{c}$ that makes its nth power countably compact and its $(n+1)$st power not countably compact.

Proof. Since $n \geqslant 2$, given two topologies as in Example 2.2, their squares are countably compact but their products are not, thus they are not homeomorphic. $\square$

Throughout the construction, n will be a fixed positive integer. The groups will be indexed by a function in $D^{\mathfrak{c}} := \{0\}^{\omega} \times 2^{\mathfrak{c}\setminus\omega}$. The topological groups will be denoted by G_f where $f \in D^{\mathfrak{c}}$.

We will use $\mathcal{U}$-limits, a concept introduced by Bernstein [1], that is useful in the study of countable compactness.

Definition 2.5. Fixed a free ultrafilter $\mathcal{U}$ on ω, we say that $x \in X$ is the $\mathcal{U}$-limit point of a sequence $\{x_m\colon m \in \omega\} \subseteq X$ if $\{m \in \omega\colon x_m \in U\} \in \mathcal{U}$ for every neighbourhood U of x.

The basic facts on $\mathcal{U}$-limit points that will be used in this paper are the following:

Lemma 2.6. *Assume that all spaces below are Hausdorff.*

(0) *The $\mathcal{U}$-limit point of a sequence is unique and for sequences in compact spaces, there exists a $\mathcal{U}$-limit point for each free ultrafilter $\mathcal{U}$.*
(1) *x is an accumulation point of $\{x_n\colon n \in \omega\}$ if and only if there exists a free ultrafilter $\mathcal{U}$ over ω such that x is the $\mathcal{U}$-limit point of this sequence.*
(2) *$\{x_\alpha\}_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ is the $\mathcal{U}$-limit point of $\{\{x_{\alpha,n}\}_{\alpha\in I}\colon n \in \omega\}$ if and only if x_α is the $\mathcal{U}$-limit point of $\{x_{\alpha,n}\colon n \in \omega\}$ for each $\alpha \in I$.*
(3) *In topological groups, if x is the $\mathcal{U}$-limit point of $\{x_n\colon n \in \omega\}$ and y is the $\mathcal{U}$-limit point of $\{y_n\colon n \in \omega\}$ then $x + y$ is the $\mathcal{U}$-limit point of $\{x_n + y_n\colon n \in \omega\}$.*

3. The sketch and some auxiliary lemmas

We assume that $2^{<\mathfrak{c}} = \mathfrak{c}$. Given a family of functions $\{z_t\colon t \in F\}$ with $z_t \in 2^{\beta}$ for some fixed β and F finite, we denote by z_F the sum $\sum_{t\in F} z_t$.

Define $D^{\alpha} = \{0\}^{\omega} \times 2^{\alpha\setminus\omega}$ if $\alpha \geqslant \omega$ and $D^m = \{0\}^m$ if m is a natural number. The set $D^{<\alpha}$ is defined as $\bigcup_{\beta<\alpha} D^{\beta}$.

We will construct $2^{<\mathfrak{c}}$ linearly independent elements of $2^{\mathfrak{c}}$ which will be denoted by $x_{f,j}$ where $f \in D^{\alpha}$ for some $\alpha < \mathfrak{c}$ and $j < n+2$. If f is the unique element of D^m, we will also denote the function by $x_{m,j}$.

Given $f \in D^{\mathfrak{c}}$ we define $G_f = \langle\{x_{f|_\alpha,j}\colon \alpha < \mathfrak{c} \wedge j < n+2\}\rangle$. Each G_f will satisfy the conditions of Lemma 2.1 and it will not contain any non-trivial convergent sequences. Furthermore, for distinct $f, h \in D^{\mathfrak{c}}$, the set $G_f \cap G_h$ is an infinite subgroup of size $< \mathfrak{c}$ thus, $G_f \times G_h$ is not countably compact.

The countable compactness of the nth power of each G_f will be attained by producing accumulation points for a number of sequences in a product of at most n copies of G_f. These sequences will be related to some kind of linear independence for sequences of k-tuples for k at most n.

We fix the following enumeration to obtain the conditions of Lemma 2.1.

Definition 3.1. Let $\{\mathcal{K}_\alpha\colon \alpha < \mathfrak{c},\ \alpha \text{ even}\}$ be an enumeration of all functions $\mathcal{K}\colon n+1 \to [D^{<\mathfrak{c}} \times (n+2)]^{<\omega}$ and $\{\mathcal{K}_\alpha\colon \alpha < \mathfrak{c}, \alpha \text{ odd}\}$ be an enumeration of all functions $\mathcal{K}\colon n \cup \{n+1\} \to [D^{<\mathfrak{c}} \times (n+2)]^{<\omega}$ so that $\bigcup_{i \in \operatorname{dom} \mathcal{K}_\alpha} \mathcal{K}_\alpha(i) \subseteq D^{<\alpha} \times (n+2)$ for every $\alpha \in [\omega, \mathfrak{c})$.

The proof of the Lemma below is straightforward and it is left to the reader.

Lemma 3.2. *If for every α even $\{m \in \omega\colon x_{m,j}(\alpha) = x_{\mathcal{K}_\alpha(j)}(\alpha) \forall j \in n+1\}$ is finite and for every α odd $\{m \in \omega\colon x_{m,j}(\alpha) = x_{\mathcal{K}_\alpha(j)}(\alpha) \forall j \in n \cup \{n+1\}\}$ is also finite, then the set T of Lemma* 2.1 *is closed and discrete in $(G_f)^{n+1}$, for each $f \in D^{\mathfrak{c}}$.*

The following enumeration will be used to obtain countable compactness in the product and the non-existence of non-trivial convergent sequences.

Notation. Let $\{\mathcal{F}_\alpha\colon \omega \leqslant \alpha < \mathfrak{c}\}$ be an enumeration of all functions $\mathcal{F}\colon S \times \omega \to [D^{<\mathfrak{c}} \times (n+2)]^{<\omega}$ such that $S \in [n+2]^{\leqslant n} \setminus \{\emptyset\}$ and for every $F \subseteq D^{<\mathfrak{c}} \times (n+2)$ finite, the family $\{\{(g,j)\}\colon (g,j) \in F\} \cup \{\mathcal{F}(i,m)\colon i \in S\}$ is linearly independent for all but finitely many $m \in \omega$. We assume that $\bigcup_{(i,m) \in \operatorname{dom} \mathcal{F}_\alpha} \mathcal{F}_\alpha(i,m) \subseteq D^\alpha \times (n+2)$. Denote by S_α the unique subset of $n+2$ such that $\operatorname{dom} \mathcal{F}_\alpha = S_\alpha \times \omega$.

Lemma 3.3. ($\mathrm{MA}_{\text{countable}}$) *Fix $h \in D^{\mathfrak{c}}$. Suppose that $(x_{h|_\alpha,i})_{i \in S_\alpha}$ is an accumulation point of $\{(x_{\mathcal{F}_\alpha(i,m)})_{i \in S_\alpha}\colon m \in \omega\}$ for every $\alpha \in [\omega, \mathfrak{c})$. Then the n-th power of G_h is countably compact and G_h has no non-trivial convergent sequences.*

Proof. Let $\{(a_{i,m})_{i<n}\colon m \in \omega\}$ be any sequence in the n-th power of G_h.

Claim 1. There exists $c_i \in G_h$ for each $i < n$, $S \subseteq n$, $E_i \subseteq S$ for each $i < n$ and $A \subseteq \omega$ infinite such that $\{(a_{i,m})_{i \in S}\colon m \in A\}$ has an accumulation point in $(G_h)^S$ and $c_i = a_{i,m} - \sum_{j \in E_i} a_{j,m}$ for each $m \in A$.

The proof of Claim 1 is more technical, so we will first apply Claim 1 and prove it after Claim 2.

Claim 2. The sequence $\{(a_{i,m})_{i<n}\colon m \in \omega\}$ has an accumulation point in $(G_h)^n$.

Proof. Let $S \subseteq n$, $E_i \subseteq S$ for each $i < n$ and $A \subseteq \omega$ as in Claim 1. Let $(b_i)_{i \in S} \in (G_h)^S$ be an accumulation point of the sequence $\{(a_{i,m})_{i \in S}\colon m \in A\}$. Then there exists a free ultrafilter p on ω such that $b_i = p\text{-}\lim\{a_{i,m}\colon m \in \omega\}$ for each $i \in S$. Then, for each

$i < n$, we have $p\text{-}\lim\{a_{i,m}\colon m \in \omega\} = c_i + p\text{-}\lim\{\sum_{j\in E_i} a_{j,m}\colon m \in \omega\} = c_i + \sum_{j\in E_i} p\text{-}\lim\{a_{j,m}\colon m \in \omega\} = c_i + \sum_{j\in E_i} b_j$. Thus, each sequence $\{a_{i,m}\colon m \in \omega\}$ has a p-limit point in G^h. Therefore, the sequence $\{(a_{i,m})_{i<n}\colon m \in \omega\}$ has an accumulation point in $(G_h)^n$.

Proof of Claim 1. Let q be a selective ultrafilter on ω (such ultrafilter exist under $\mathrm{MA}_{\mathrm{countable}}$ and we will mention the necessary properties in due time). We can define the following relation in $(G_h)^\omega$: two functions f and g in $(G_h)^\omega$ are equivalent if $\{n \in \omega\colon f(n) = g(n)\} \in q$. Let $[f]$ denote the set of equivalent elements of f in $(G_h)^\omega$ and let $((G_h)^\omega)/q$ be the set of equivalent classes. Given two classes $[f]$ and $[g]$, denote by $[f]+[g]$ the class $[f+g]$. Then $((G_h)^\omega)/q$ with this operation is a vector space over 2, since G_h is a vector space over 2. Given an element $c \in G$, denote by $[c]$ the class of the constant function c.

Now, let R be the vector subspace generated by $\{[\{a_{i,m}\colon m \in \omega]\colon i < n\}$ and $C \subseteq G_h$ be such that $\{[c]\colon c \in C\}$ is a basis for $R \cap \{[c]\colon c \in G_h\}$. Let $S \subseteq n$ be such that

(I) $\{[\{a_{i,m}\colon m \in \omega\}]\colon i \in S\} \cup \{[c]\colon c \in C\}$ is a basis for R.
For each $i < n$, let $E_i \subseteq S$ and $C_i \subseteq C$ be such that $[\{a_{i,m}\colon m \in \omega\}] = \sum_{c\in C_i}[c] + \sum_{j\in E_i}[\{a_{i,m}\colon m \in \omega\}]$ for each $i < n$. Then clearly, $\{a_{i,m}\colon m \in \omega\} - \sum_{j\in E_i}\{a_{i,m}\colon m \in \omega\}$ is equivalent to a constant sequence. Thus, there exists $B \in q$ and $c_i \in G_h$ for each $i < n$ such that

(II) $c_i = \sum C_i = a_{i,m} - \sum_{j\in E_i} a_{i,m}$ for each $i < n$ and $m \in B$.
We claim that

(III) $\sum_{i\in T}[\{a_{i,m}\colon m \in \omega\}] \notin \{[d]\colon d \in G\}$ for each $T \subseteq S$ non-empty.

Indeed, let $D \subseteq G$ be such that $D \supseteq C$ and D is a basis of G. Then, from (I), it follows that $\{[\{a_{i,m}\colon m \in \omega\}]\colon i \in S\} \cup \{[c]\colon c \in D\}$ is linearly independent. Therefore, $\langle\{[\{a_{i,m}\colon m \in \omega\}]\colon i \in S\}\rangle \cap \langle\{[c]\colon c \in D\}\rangle = \{[0]\}$ and (III) holds.

We claim that there exists $A \in q$ with $A \subseteq B$ such that

(IV) $\{\sum_{i\in T} a_{i,m}\colon m \in A\}$ is faithfully indexed for each $T \subseteq S$ non-empty.

Indeed, since q is selective, there exists $A_T \in p$ such that $\{\sum_{i\in T} a_{i,m}\colon m \in A_T\}$ is either constant or faithfully indexed. It follows from (III) that $\{\sum_{i\in T} a_{i,m}\colon m \in A_T\}$ cannot be constant, thus, it is faithfully indexed. Define $A = B \cap \bigcap_{\emptyset\neq T\subseteq S} A_T \in q$. Then A is as required.

Let $\{m_k\colon k \in \omega\}$ be an enumeration of A. Let $\mathcal{F}\colon S \times \omega \to [D^{<\mathfrak{c}} \times (n+2)]^{<\omega}$ be such that $x_{\mathcal{F}(i,k)} = a_{i,m_k}$ for each $i \in S$ and $k \in \omega$.

We claim that there exists $\alpha < \mathfrak{c}$ such that $\mathcal{F}_\alpha = \mathcal{F}$. For that, it suffices to show that if $F \subseteq D^{<\mathfrak{c}} \times (n+2)$ finite, then the family $\{\{(c,j)\}\colon (c,j) \in F\} \cup \{\mathcal{F}(i,k)\colon i \in S\}$ is linearly independent for all but finitely many $k \in \omega$. Suppose by contradiction that this is not the case. Then there exists $E \subseteq F$, $\emptyset \neq T \subseteq S$ and $K \subseteq \omega$ infinite such that $\Delta\{\mathcal{F}(i,k)\colon i \in T\} = E$ for each $k \in K$.

Then, $\sum_{i\in T} a_{i,m_k} = x_E$ for each $k \in K$. However, this contradicts (IV). Now, by hypothesis, the sequence $\{(x_{\mathcal{F}(i,k)})_{i\in S}\colon k \in \omega\}$ has an accumulation point in $(G_h)^S$. We are done with Claim 1, since the last sequence is a subsequence of $\{(a_{i,m})_{i\in S}\colon m \in \omega\}$. $\square$

We will show now that G_h has no non-trivial convergent sequences. Let $\{b_m\colon m \in \omega\}$ be a non-trivial sequence in the group G_h. We can assume without loss of generality that this sequence is an injection and that none of its elements is the identity in G_h. Then, there exists $\alpha_i \in [\omega, \mathfrak{c})$ such that $S_{\alpha_i} = \{0\}$ and $b_{2m+i} = x_{\mathcal{F}_{\alpha_i}(0,m)}$ for every $m \in \omega$ and $i < 2$. Therefore, $\{b_m\colon m \in \omega\}$ has two accumulation points $x_{h|_{\alpha_0},0}$ and $x_{h|_{\alpha_1},0}$ in G_h. Since G_h is Hausdorff, the sequence $\{b_m\colon m \in \omega\}$ does not converge. $\square$

4. The inductive hypothesis and the partial order

The construction of the linearly independent set $\{x_{f,i}\colon f \in D^{<\mathfrak{c}} \wedge i < n+2\}$ is by induction. It is left to the reader to check that if the inductive hypothesis below are satisfied and $x_{f,i} = \bigcup_{\alpha<\mathfrak{c}} x_{f,i}|_\alpha$ for each $f \in D^{<\mathfrak{c}}$ and $i < n+2$ then the conditions of Lemmas 2.1 and 3.3 are satisfied for the groups $G_h = \langle\{x_{h|_\alpha,i}\colon \alpha < \mathfrak{c},\ i < n+2\}\rangle$, where $h \in 2^{\mathfrak{c}}$.

At an infinite stage α we will have defined $\{x_{f,i}|_\alpha\colon f \in D^\alpha\}$ such that the following are satisfied:

(1) $x_{f,i}|_\beta \in 2^\beta$ for each $\beta \leqslant \alpha$ and $f \in D^{\leqslant\alpha}$;
(2) $x_{f,i}|_\beta \subseteq x_{f,i}|_\alpha$ for each $\beta < \alpha$ and $f \in D^{\leqslant\beta}$;
(3) if $\omega \leqslant \beta < \alpha$ and $\mathcal{K}_\beta(0) \neq \emptyset$ then $x_{\mathcal{K}_\beta(0)}(\beta) \neq 0$;
(4) if $\omega \leqslant \beta \leqslant \alpha$ and $f \in D^\beta$ then $\{x_{f,i}|_\alpha\}_{i\in S_\beta}$ is an accumulation point of the sequence $\{\{x_{\mathcal{F}_\beta(m,i)}|_\alpha\}_{i\in S_\beta}\colon m \in \omega\}$;
(5) the set $\{(x_{m,n}|_\alpha, x_{m,n+1}|_\alpha)\colon m \in \omega\}$ is dense in $2^\alpha \times 2^\alpha$;
(6) if $\omega \leqslant \beta < \alpha$ is even then $\{m \in \omega\colon x_{m,i}(\beta) = x_{\mathcal{K}_\beta(i)}(\beta)\ \forall i \in n+1\}$ is finite;
(7) if $\omega \leqslant \beta < \alpha$ is odd then $\{m \in \omega\colon x_{m,i}(\beta) = x_{\mathcal{K}_\beta(i)}(\beta)\ \forall i \in n \cup \{n+1\}\}$ is finite.

At stage ω, only condition (5) is not trivially satisfied. Choose $x_{m,i}|_\omega \in 2^\omega$ for each $m \in \omega$ and $i < n+2$ such that the sequence $\{\{x_{m,i}\}_{i<n+2}\colon m \in \omega\}$ is dense in $(2^\omega)^{n+2}$.

At limit stage α, define $x_{f,i}|_\alpha = \bigcup_{\operatorname{dom} f<\gamma<\alpha} x_{f,i}|_\gamma$ for each $f \in D^{<\alpha}$ and $i < n+2$. Clearly, all seven inductive hypothesis will be satisfied.

At successor stage $\alpha = \gamma+1$, define $\{x_{f,i}|_\gamma\}_{i<n+2}$ for every $f \in 2^\gamma$ so that $\{x_{f,i}|_\gamma\}_{i\in S_\gamma}$ is an accumulation point of the sequence $\{\{x_{\mathcal{F}_\gamma(i,m)}|_\gamma\}_{i\in S_\gamma}\colon m \in \omega\}$. We will assume that α is odd (thus, γ is even), as the other case is similar.

As in other constructions of countably compact groups without non-trivial convergent sequences (see [16,10,19]), some form of Martin's Axiom will be used to split many subsequences of a sequence at a successor stage.

In order to apply $\mathrm{MA}_{\text{countable}}$ rather than MA in the successor stage, we use the following definition (compare with [19]):

Definition 4.1. For each $\xi < \alpha$ define by induction a set $\mathcal{I}_\xi$ as ω if $\xi \leqslant \omega$ and $\bigcup\{\mathcal{I}_{\operatorname{dom} f} \cup \{f\}\colon (\{f\} \times (n+2)) \cap (\bigcup_{m\in\omega\wedge i\in S_\xi} \mathcal{F}_\xi(i,m)) \neq \emptyset\}$.

An easy induction leads to the following result:

Lemma 4.2. *The set $\mathcal{I}_\xi$ is countable for each $\xi < \alpha$.*

We define now the partial order for the successor stage α.

Throughout the remainder of this section, let $\mathcal{I} = \bigcup_{(f,j)\in\bigcup_{i<n+1}\mathcal{K}_\gamma(i)} \mathcal{I}_{\operatorname{dom} f}$ and fix a function r whose domain is $F \times (n+1)$, where F is a finite subset of $\mathcal{I}$, $\operatorname{dom} r \supseteq \bigcup_{i<n+2}\mathcal{K}_\gamma(i)$ and $F \cap \omega \in \omega$. If $\mathcal{K}_\gamma(0) \neq \emptyset$, choose r so that $\sum_{(f,i)\in\mathcal{K}_\gamma(0)} r(f,i) = 1$.

With the aid of $\mathrm{MA}_{\mathrm{countable}}$, we will construct a function $\psi : \mathcal{I} \times (n+2) \to 2$ and define $x_{f,i}(\gamma) = \psi(f,i)$ for each $(f,i) \in \mathcal{I} \times (n+2)$ so that the inductive conditions restricted to $\mathcal{I} \times (n+2)$ are satisfied. The definition of $x_{f,i}(\gamma)$ for $(f,i) \in (\alpha \setminus \mathcal{I}) \times (n+2)$ will be done by induction later and do not require $\mathrm{MA}_{\mathrm{countable}}$.

Definition 4.3. Let $\mathbb{P}$ be a partial order whose underlying set is the family of all functions $p : F \times (n+2) \to 2$ with $F \in [\mathcal{I}]^{<\omega}$ and $F \cap \omega \in \omega$. Let M_p be the unique integer such that $F \cap \omega = M_p$. Given $p, q \in \mathbb{P}$, define $p \leqslant q$ if and only if $p \supseteq q$ and for each $k \in [M_q, M_p)$ there exists $i < n+1$ such that $p(k,i) \neq \sum_{(g,j)\in\mathcal{K}_\gamma(i)} r(g,j)$.

The dense sets defined below will be used to define $x_{f,i}(\gamma)$ for each $f \in \mathcal{I}$ and $i < n+2$. The proof of the next lemma is left to the reader.

Lemma 4.4. *The set* $\{p \in \mathbb{P}\colon \operatorname{dom} p \supseteq \{f\} \times (n+2)\}$ *is dense in* $\mathbb{P}$ *for each* $f \in \mathcal{I}$.

The following sets will be used to define the dense sets needed for condition 4.

Definition 4.5. For each $f \in \mathcal{I}$ and $F \subseteq \gamma$ finite, define $\mathcal{E}(f,F) = \{m \in \omega\colon \forall i \in S_{\operatorname{dom} f} (\sum_{(g,j)\in\mathcal{F}_{\operatorname{dom} f}(i,m)} x_{g,j}|_F = x_{f,i}|_F)\}$.

Note that by hypothesis, the sets defined above are infinite.

Lemma 4.6. *The set* $\mathcal{D}(f,F,v,M) = \{p \in \mathbb{P}\colon \exists m \in \mathcal{E}(f,F) \setminus M$ *such that* $\operatorname{dom} p \supseteq \bigcup_{i\in S_\beta}\mathcal{F}_\beta(i,m)$ *and* $(\forall i \in S_\beta)\sum_{(g,j)\in\mathcal{F}_\beta(i,m)} p(g,j) = v(i)\}$ *is dense for each* $f \in \mathcal{I}$, $F \in [\gamma]^{<\omega}$, $v \in 2^{S_\beta}$ (*where* $\beta = \operatorname{dom} f$) *and* $M \in \omega$.

Proof. Fix f, β, F, v and M as above and let q be an arbitrary element of $\mathbb{P}$. Set $E = \{\{(g,j)\}\colon (g,j) \in \operatorname{dom} q\}$. By hypothesis, there exists $m \in \mathcal{E}(\beta, F) \setminus M$ such that $E \cup \{\mathcal{F}_\beta(i,m)\colon i \in S_\beta\}$ is linearly independent.

Let $N \in \omega$ be such that $(\operatorname{dom} q \cup \bigcup\{\mathcal{F}_\beta(i,m) : i \in S_\beta\}) \cap (\omega \times (n+2)) \subseteq N \times (n+2)$. We will define p such that $M_p = N$.

We claim that there exists $k_t \in n+1$ for each $t \in [M_q, N)$ such that $E \cup \{\mathcal{F}_\beta(i,m)\colon i \in S_\beta\} \cup \{\{(t,k_t)\}\colon t \in [M_q, N)\}$ is linearly independent. Indeed, $E \cup \{\{(M_q, j)\}\colon j < n+1\}$ is a linearly independent set of size strictly bigger than $E \cup \{\mathcal{F}_\beta(i,m)\colon i < S_\beta\}$. Therefore, there exists $k_{M_q} \in n+1$ such that $E \cup \{\mathcal{F}_\beta(i,m)\colon i \in S_\beta\} \cup \{(M_q, k_{M_q})\}$ is linearly independent. Now, $E \cup \{(M_q+1, j)\}\colon j < n+1\} \cup \{(M_q, k_{M_q})\}$ is a linearly independent set of size strictly bigger than $E \cup \{\mathcal{F}_\beta(i,m)\colon i \in S_\beta\} \cup \{(M_q, k_{M_q})\}$. Proceeding by induction, one can find $k_t \in n+1$ for each $t \in [M_q, N)$ as required.

Define $\phi : E \cup \{\mathcal{F}_\beta(i,m)\colon i \in S_\beta\} \cup \{\{(t,k_t)\}\colon t \in [M_q, N)\} \to 2$ as $\phi(\{(g,j)\}) = q(g,j)$ if $\{(g,j)\} \in E$; $\phi(\mathcal{F}_\beta(i,m)) = v(i)$ if $i \in S_\beta$ and $\phi(\{(t,k_t)\}) = 1 -$

$\sum_{(g,j)\in\mathcal{K}_\gamma(i)} r(g,j)$ if $t\in[M_q,N)$. Extend ϕ to a homomorphism $\bar\phi$ whose domain is $[\mathcal{I}\times(n+2)]^{<\omega}$ and its range is 2. Define $p(g,j)=\bar\phi(\{(g,j)\})$, for every $(g,j)\in \operatorname{dom} q\cup\bigcup\{\mathcal{F}_\beta(i,m)\colon i\in S_\beta\}\cup(N\times(n+2))$. Then p is an extension of q belonging to $\mathcal{D}(f,F,v,M)$. $\square$

The dense sets below will be needed for condition (5).

Definition 4.7. For each $A\in[\gamma]^{<\omega}$, $t\in 2^A\times 2^A$ and $(a_0,a_1)\in 2\times 2$, let $\mathcal{C}(t,a_0,a_1,M)=\{p\in\mathbb{P}\colon \exists m\in\omega\setminus M\colon (x_{m,n}|_A, x_{m,n+1}|_A)=t$ and $p(m,n+j)=a_j$ for $j<2\}$.

Lemma 4.8. *The sets defined above are dense in* $\mathbb{P}$.

Proof. Similar to the proof of Lemma 4.6. $\square$

We are now ready to finish the construction. Applying $\mathrm{MA}_{\text{countable}}$, there exists a filter $\mathbb{G}$ over $\mathbb{P}$ which intercepts every dense set defined in Lemmas 4.4, 4.6 and 4.8. As $\mathbb{G}$ intercepts the dense sets in Lemma 4.4, the set $\psi=\bigcup\mathbb{G}$ is a function from $\mathcal{I}\times(n+2)$ into 2. Define $x_{f,i}(\gamma)=\psi(f,i)$ for every $(f,i)\in\mathcal{I}\times(n+2)$.

Clearly conditions (1) and (2) are satisfied for each $f\in\mathcal{I}$ and $i\in(n+2)$. Condition (3) is satisfied by the choice of r and the fact that ψ extends r. We leave for the reader to check that condition (4) for $f\in\mathcal{I}$ and condition (5) follow from the fact that $\mathbb{G}$ intercepts the dense sets in Lemmas 4.6 and 4.8, respectively.

We will check condition (6). Let p be an element of $\mathbb{G}$. Then, for each $m\in\omega\setminus M_p$ (M_p is defined with the partial order $\mathbb{P}$), there exists $q\in\mathbb{G}$ such that $q\leqslant p$ and $m\in M_q$. Then, by the definition of the ordering, there exists $i<n+1$ such that $x_{m,i}(\gamma)=q(m,i)\neq\sum_{(g,j)\in\mathcal{K}_\gamma(i)} r(g,j)=x_{\mathcal{K}_\gamma(i)}(\gamma)$. Thus, the set in condition (6) for $\beta=\gamma$ is contained in M_p, therefore, is finite. Condition (7) is trivially satisfied at this stage, since γ is even. At stage α even, the proof of condition (7) is similar to the proof of condition (6).

We still have to define $x_{f,i}(\gamma)$ for $f\in D^\gamma\setminus\mathcal{I}$ and $i<n+2$. Let η be the least ordinal for which $x_{f,i}(\gamma)$ has not yet been defined for some $f\in D^\eta$. For each such an f, fix an ultrafilter $\mathcal{U}$ over ω such that $(x_{f,i}|_\gamma)_{i\in S_\eta}$ is the $\mathcal{U}$-limit of the sequence $\{(x_{\mathcal{F}_\eta(i,m)}|_\gamma)_{i\in S_\eta}\colon m\in\omega\}$. Define $x_{f,i}(\gamma)$ arbitrarily if $i\in(n+2)\setminus S_\eta$ and $x_{f,i}(\gamma)=\mathcal{U}$-limit $\{(x_{\mathcal{F}_\eta(i,m)}(\gamma)\colon m\in\omega\}$ if $i\in S_\eta$. Clearly conditions (1), (2) and (4) are now satisfied by any $(f,i)\in D^\eta\times(n+2)$.

5. $\mathcal{G}$-spaces and finite products

Modifying a game due to Gruenhage [15], Bouziad [3] defined the following game:

Let X be a space and $x\in X$. The game $\mathcal{G}(x,X)$ is played as follows: there are two players (I) and (II). Player (I) plays first and chooses a neighborhood U_0 of x. Player (II) then responds by choosing $x_0\in U_0$. After choosing $U_0,\dots,U_n$ and points $x_i\in U_i$ for each $i\leqslant n$, player (I) chooses a neighborhood U_{n+1} of x and player (II) chooses $x_{n+1}\in U_{n+1}$.

Player (I) wins if $\{x_n\colon n\in\omega\}$ has an accumulation point in X, otherwise player (II) wins.

The space X is a $\mathcal{G}$-space if player (I) has a winning strategy for each game $\mathcal{G}(x, X)$.

In [12], the authors showed that for each $n \in \omega$ there exists a space X such that X^n is a $\mathcal{G}$ space but X^{n+1} is not. They asked whether there is such an example that it is a topological group. We give a consistent answer using Martin's Axiom.

We use the following lemma:

Lemma 5.1. *Let $\{x\} \cup \{x_n\colon\ n \in \omega\} \subseteq X$ and $\{y\} \cup \{y_n\colon\ n \in \omega\} \subseteq Y$ such that x is an accumulation point of $\{x_n\colon\ n \in \omega\}$ and y is an accumulation point of $\{y_n\colon\ n \in \omega\}$. If for every injections $\psi, \phi : \omega \to \omega$ with $\phi"[\omega] \cap \psi"[\omega] = \emptyset$, the sequence $\{(x_{\psi(k)}, y_{\phi(k)})\colon\ k \in \omega\}$ is closed discrete then player* (II) *has a winning strategy for the game $\mathcal{G}((x, y), X \times Y)$. In particular, the product $X \times Y$ is not a $\mathcal{G}$-space.*

Proof. Without loss of generality, player (I) chooses neighbourhoods U_n and V_n of x and y, respectively, at the nth move. Suppose that player (II) has chosen $k_0, \ldots, k_{n-1}$ and $l_0, \ldots, l_{n-1}$ pairwise distinct such that $(x_{k_i}, y_{k_i}) \in U_i \times V_i$ for each $i \leqslant n$. Player (II) chooses $k_n \in \omega \setminus \{k_0, \ldots, k_{n-1}, l_0, \ldots, l_{n-1}\}$ and $l_n \in \omega \setminus \{l_0, \ldots, l_{n-1}, k_0, \ldots, k_n\}$ such that $(x_{k_n}, y_{l_n}) \in U_n \times V_n$. By hypothesis, the sequence $\{(x_{k_n}, y_{l_n})\colon\ n \in \omega\}$ is closed discrete. Therefore, player (II) has a winning strategy and $X \times Y$ is not a $\mathcal{G}$-space. $\square$

Example 5.2. ($\mathrm{MA}_{\mathrm{countable}}$) For each positive integer n, there exists a topological group G such that G^n is a $\mathcal{G}$-space but G^{n+1} is not.

Proof. Fix $f \in 2^{\mathfrak{c}}$. In the construction of Example 2.2, we can make the sequence $\{\{x_{m,j}\colon\ j < n\}\colon\ m \in \omega\}$ dense in $(2^{\mathfrak{c}})^n$. There are $\mathfrak{c}$ many pairs of injections $\psi, \phi : \omega \to \omega$ whose ranges are disjoint, thus, we can modify the partial orders at successor stages to make all the sequences $\{\{x_{\psi(m),j}\colon\ j < n\} \cup \{x_{\phi(m),n}\}\colon\ m \in \omega\}$ closed and discrete in $(G_f)^{n+1}$.

For example, for a fixed pair ϕ and ψ of injective function with disjoint range and a function $r : E \times (n+2) \to 2$ with $\phi^{-1}(\omega \cap E) = \psi^{-1}(\omega \cap E) \in \omega$, let $\mathbb{P}$ be a partial order whose underlying set is the family of all functions $p : F \times (n+2) \to 2$ with $F \in [\mathcal{I}]^{<\omega}$ and $\phi^{-1}(\omega \cap F) = \psi^{-1}(\omega \cap F) = M_p \in \omega$. Given $p, q \in \mathbb{P}$, define $p \leqslant q$ if and only if $p \supseteq q$ and for each $k \in [M_q, M_p)$ either there exists $i < n$ such that $p(\psi(k), i) \neq \sum_{(g,j) \in \mathcal{K}_\gamma(i)} r(g, j)$ or $p(\phi(k), n) \neq \sum_{(g,j) \in \mathcal{K}_\gamma(n)} r(g, j)$.

Applying Lemma 5.1 above, $(G_f)^{n+1}$ is not a $\mathcal{G}$-space.

The product $(G_f)^n$ is countably compact, therefore, a $\mathcal{G}$-space. $\square$

Note added in September 2003

Recently, the authors of [13] showed the existence of two countably compact groups whose product is not countably compact from the existence of a selective ultrafilter and asked whether there is a countably compact group whose square is not countably compact from the existence of a selective ultrafilter.

We can extend their question to finite powers:

Question 5.3. *Does the existence of a selective ultrafilter implies, for each $n \in \omega$, the existence of a countably compact topological group G such that G^n is countably compact but G^{n+1} is not countably compact*?

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